Math 436 midterm (practice)

Name: ____________________

This exam has 8 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (10 pts)

(a) State the definition of a topology on a set X.

> **Solution:** Omitted. See Definition 2.1 in the textbook.

(b) Find a family of open subsets of the real line $\mathbb{R}$ whose intersection is not open.

> **Solution:** Consider the family $\{A_n\}_{n\in\mathbb{N}_+}$, where $A_n = (-\frac{1}{n}, \frac{1}{n})$. We have
>
> $$\bigcap_{n=1}^{\infty} A_n = \{0\}$$
>
> which is not open.

Question 2. (10 pts)

(a) State the definition of compactness.

> **Solution:** Omitted. See Definition 3.2 in the textbook.

(b) Is it possible for a discrete space to be compact? Explain.

> **Solution:** Yes, for example, consider the discrete space $X = \{a, b\}$ of two points. In this case, in fact, any open cover of X can only consist of finitely many members, so automatically has a finite subcover.

Question 3. (10 pts)

Let $f\colon X \to \mathbb{R}$ be a continuous function on a topological space X. Suppose U is an open set of X, is $f(U)$ always open in $\mathbb{R}$? Explain.

Solution: $f(U)$ is not always open. For example, let $X = [0, 1]$ and the constant map $f\colon [0, 1] \to \mathbb{R}$ by $f(x) \equiv 0$. Choose $U = X = [0, 1]$ which is open in X, but $f(U) = \{0\}$ is not open in $\mathbb{R}$.

Question 4. (10 pts)

Suppose X is compact space and $f: X \to \mathbb{R}$ is a continuous real valued function on X. If $f(x) > 0$ for all $x \in X$, prove that there exists a number $r > 0$ such that $f(x) > r$ for all $x \in X$.

Solution: Since X is compact and $f: X \to \mathbb{R}$ is continuous, f attains its minimum. That is, there exists $x_0 \in X$ such that $f(x_0) = \inf(f)$. By assumption, $f(x_0) > 0$. Let $r = f(x_0)/2$. Then

$$f(x) \geq f(x_0) > f(x_0)/2 = r$$

for all $x \in X$. This finishes the proof.

Question 5. (15 pts)

Let (X, d_X) and (Y, d_Y) be two metric spaces. Suppose $f\colon X \to Y$ is a map such that $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

(a) Prove that f is injective.

> **Solution:** If $x_1 \neq x_2$ in X, then
> $$d_X(x_1, x_2) > 0.$$
> It follows that
> $$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) > 0.$$
> Therefore $f(x_1) \neq f(x_2)$ in Y. This proves that f is injective.

(b) Prove that f is continuous.

> **Solution:** Let β be the collection of all open balls in Y, that is,
> $$\beta = \{B_r(y) \mid \forall r > 0 \text{ and } \forall y \in Y\}.$$
> Note that β is a base of the topology of Y. It suffices to show that the inverse image of any $B_r(y)$ is open.
> Suppose $x \in f^{-1}(B_r(y))$, that is, $d_Y(f(x), y) < r$. We want to show that there is an open ball $B_\varepsilon(x)$ centered at x of radius ε such that $B_\varepsilon(x) \subset f^{-1}(B_r(y))$. Since $d_Y(f(x), y) < r$, we can choose $\varepsilon > 0$ such that $d_Y(f(x), y) < r - \varepsilon$. Then for any $x' \in B_\varepsilon(x)$, we have
> $$\begin{aligned} d_Y(f(x'), y) &\leq d_Y(f(x'), f(x)) + d_Y(f(x), y) \\ &= d_X(x', x) + d_Y(f(x), y) \\ &< \varepsilon + (r - \varepsilon) = r \end{aligned}$$
> Therefore, $x' \in B_\varepsilon(x) \Rightarrow f(x') \in B_r(y)$. In other words, $B_\varepsilon(x) \subset f^{-1}(B_r(y))$. We have shown that for any $x \in f^{-1}(B_r(y))$, there exists an open ball centered at x which is contained in $f^{-1}(B_r(y))$. This proves that $f^{-1}(B_r(y))$ is open.

Question 6. (10 pts)

Let X be a discrete topological space with at least two distinct points. Show that X is not connected.

Solution: Choose a point $x \in X$. Define $A = \{x\}$ and $B = A^c = X - \{x\}$. Since X has at least two distinct points, both A and B are not empty. Since X is a discrete space, both A and B are open. It follows that X is the union of two disjoint nonempty open sets. Therefore X is not connected.

Question 7. (15 pts)

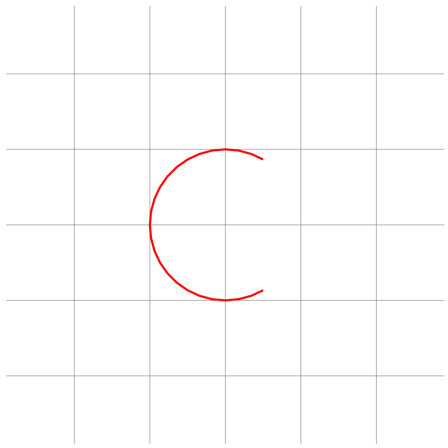
(a) Let A and B be two connected subsets of $\mathbb{R}$. If $A \cap B \neq \emptyset$, show that $A \cap B$ is connected.

(b) Find two connected subsets C and D of $\mathbb{R}^2$ such that $C \cap D \neq \emptyset$ and $C \cap D$ is *not* connected.

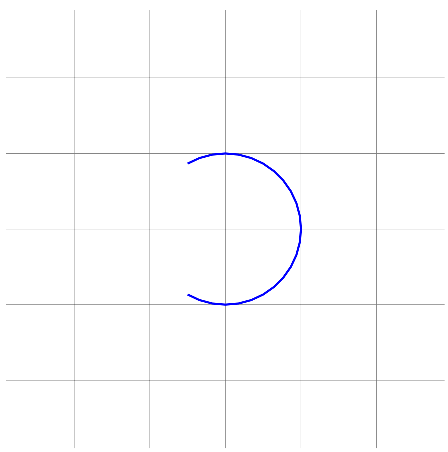
Solution:

(a) Since A and B are connected subsets of $\mathbb{R}$, A and B are both intervals. By assumption, $A \cap B \neq \emptyset$. Then $A \cap B$ is also an interval, hence connected.

(b) Let C be the arc on the unit circle starting at angle $\pi/3$ and ending at $5\pi/3$.



Let D be the arc on the unit circle starting at angle $-2\pi/3$ and ending at $2\pi/3$.



Question 8. (20 pts)

Let $f\colon [0,1] \to \mathbb{R}$ be a continuous real-valued function on $[0,1]$. The graph $G(f)$ of f is defined to be the following subset of $\mathbb{R}^2$:

$$\big\{(x, f(x)) \in \mathbb{R}^2 \mid x \in [0,1]\big\}.$$

(a) Show that $G(f)$ is compact.

> **Solution:** Define a map $h\colon [0,1] \to \mathbb{R}^2$ by
>
> $$h(x) = (x, f(x)).$$
>
> Let $p_1\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection to the first coordinate and $p_2\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection to the second coordinate. Then $p_1 \circ h(x) = x$ and $p_2 \circ h(x) = f(x)$. Both $p_1 \circ h$ and $p_2 \circ h$ are continuous. It follows that h is continuous, since $\mathbb{R}^2$ is the product space of $\mathbb{R}$ and $\mathbb{R}$.
>
> Since $[0,1]$ is compact and the continuous image of a compact space is compact, it follows $G(f) = h([0,1])$ is compact.

(b) Show that $G(f)$ is homeomorphic to $[0,1]$.

> **Solution:** $G(f)$ is a subspace of $\mathbb{R}^2$, so $G(f)$ is Hausdorff. Clearly, the map h from part (a):
>
> $$h\colon [0,1] \to G(f)$$
>
> is one-to-one, onto and continuous. By Theorem 3.7, h is a homeomorphism from $[0,1]$ to $G(f)$. In particular, this shows $G(f)$ is homeomorphic to $[0,1]$.